# STAT 231 — Lecture 22

*Bartosz Antczak* *Instructor: Michael Wallace* *November 1, 2017*

**Last Time**

We learned about the **chi-squared distribution**. One key characteristic about this distribution is that if $Z \sim G(0,1)$, then $Z^2 \sim \chi^2(1)$.

## 22.1 Section 4.6: Likelihood Intervals and Confidence Intervals

To review, we learned about likelihood intervals, and the we covered confidence intervals. Both methods are used to determine a "plausible" estimate for $\theta$.

- **Likelihood interval:** values of $\theta$ such that $R(\theta) \geq p$ for some $p$
- **Confidence interval:** values of $\theta$ such that $P[L(Y) \leq \theta \leq U(Y)] = q$

Today we'll look at how we can combine both of these formulas into one neat little package.

### 22.1.1 Likelihood Ratio Statistic

Let

$$\Lambda = -2\log\left(\frac{L(\theta)}{L(\tilde{\theta})}\right) = -2\log\left(\frac{L(\theta;Y)}{L(\tilde{\theta;Y})}\right)$$

Recall that $\tilde{\theta} = \tilde{\theta}(Y)$ is the maximum likelihood estimator of $\theta$. Because of this, $\Lambda$ is therefore a random variable that depends on $Y$. We call it the *likelihood ratio statistic*. One key characteristic about $\Lambda$ is:

$$\Lambda \sim \chi^2(1)$$

If we gather some data, we can apply this formula to determine $\theta$:

$$\lambda = -2\log R(\theta)$$

**Example 22.1.1.** *Binomial Likelihood Ratio Statistic*

$$\lambda = \frac{\theta^y(1-\theta)^{n-y}}{\left(\frac{y}{n}\right)\left(1-\frac{y}{n}\right)^{n-y}}$$

**Combining Likelihood and Confidence Intervals Together**

Observe that $0 \leq R(\theta) \leq 1$, which means that $-2\log R(\theta)$ will always be positive. To determine a confidence interval for $\Lambda$ would involve determining a value $c$ such that $\theta$ appears in $100q\%$ of the time. We define our confidence interval as

$$\{\theta \ : \ -2\log R(\theta) \leq c\} = P(W \leq c) = q$$

if $W \sim \chi^2(1)$.

But observe that

$$\{\theta \;:\; -2\log R(\theta) \leq c\} = \{\theta \;:\; R(\theta) \geq e^{-c/2}\}$$

which shows that our confidence interval can become a likelihood interval! For example, if we define a $100p\%$ likelihood interval as

$$\{\theta \;:\; R(\theta) \geq p\}$$

then this equals

$$\{\theta \;:\; -2\log R(\theta) \leq -2\log(p)\}$$

**Example 22.1.2.** *Determining both a confidence and likelihood interval*

Say we want to calculate a 95% confidence interval for $\theta$. We have

$$0.95 = P(W \leq c) = P(|Z| \leq \sqrt{c})$$

and we should know that our $c$ value is $c = 1.96^2$. To convert this to a likelihood interval, we calculate

$$\{\theta \;:\; R(\theta) \geq e^{-1.96^2/2}\} = \{\theta \;:\; R(\theta) \geq 0.147\}$$

From this we can conclude that **a 14.7% likelihood interval is equal to a 95% confidence interval.**