MATH 239 — Tutorial 3

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Deriving a Formula

Let n_t denote the number of vertices of degree t in a tree:

$$n_1 = 2 + \sum_{r=3}^{n} (r-2)n_r$$

This formula was outlined in the lecture 8 notes, subsection 8.1.1-8.1.2. Note, this formula *only* holds true for any graph that satisfies $|V(G)| = |E(G)| + 2$ (which is usually a tree).
We'll cover some questions from the course notes, in problem set 5.1.

3.1 Problem Set 5.1

3. What is the smallest number of vertices of degree 1 in a tree with 3 vertices of degree 4 and 2 vertices of degree 5?

We know that $n_4 = 3$ and $n_5 = 2$.

$$\begin{aligned} n_1 &= 2 + \sum_{r=4}^{5} n_r(r-2) && (3.1)\\ &= 2 + 3(4-2) + 2(5-2) && (3.2)\\ &= 14 && (3.3) \end{aligned}$$

Therefore, there must be at least 14 vertices of degree 1. A possible drawing of this tree is shown:

5. A cubic tree is a tree where all vertices have degree 3 or 1. Prove that a cubic tree with exactly k vertices of degree 1 has 2(k-1) vertices.

Referring to our formula again

$$\begin{aligned} n_1 &= 2 + \sum_{r=3}^{n} n_r(r-2) && (3.4)\\ k &= 2 + n_3(3-2) && (3.5)\\ n_3 &= k-2 && (3.6) \end{aligned}$$

Observe that $|V(T)| = n_1 + n_3 = (k + k - 2) = 2(k-1)$, as desired.

6. A forest is a graph with no cycles. Prove that a forest with p vertices and q edges has p-q components

Let F be a forest. Let $F = \{T_1, T_2, \cdots, T_k\}$ (where T_i is a tree for all $i \in [1, k]$). We also know that for every tree, $|V(T_i)| = 1 + |E(T_i)|$. Observe

$$\sum_{i=1}^{k} |V(T_i)| = \sum_{i=1}^{k} (1 + |E(T_i)|) \tag{3.7}$$

$$|V(F)| = k + |E(F)| \tag{3.8}$$

$$k = |V(F)| - |E(F)| \tag{3.9}$$

$$= p - q \tag{3.10}$$