MATH 239 — LECTURE 8

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Review of last lecture

We spoke about *leaves*. A leaf is a vertex of degree 1. We also covered some important theorems on trees, forests, and leaves:

Theorem 1

If G has a minimum degree of at least 2, then G contains a cycle

Theorem 2

If T is a tree and v is a leaf of T, then $T - v$ is a tree

Theorem 3

If T is a tree, then $|E(G)| = |V(G)| - 1$

8.1 Additional Theorems involving Trees

8.1.1 Corollary 1

If T is a tree on n vertices, then

$$\sum_{v \in T} \deg(v) = 2|E(T)| = 2(n-1) = 2n - 2$$

(We have $2(n-1)$ because a tree on n vertices will have $n-1$ edges, as proven in Theorem 3). Now, we can rewrite this as

$$\sum_{v \in T} (\deg(v) - 2) = \sum_{v \in T} \deg(v) - 2n = -2$$

Let d_i denote the number of vertices of degree i. This results in the sum on the left side being equal to

$$\begin{aligned}\sum_{v \in T} (\deg(v) - 2) &= \sum_{v \in T} \deg(v) - 2n \\ &= \sum_{i=1}^{n} i d_i - 2 \sum_{i=1}^{n} d_i \\ &= \sum_{i=1}^{n} (i-2) d_i\end{aligned}$$

Which means that we have:

$$\sum_{i=1}^{n} (i-2) d_i = -d_1 + 0d_2 + 1d_3 + 2d_4 + \cdots = -2$$

8.1.2 Lemma 1

If T is a tree on at least 2 vertices, then

$$d_1 = 2 + d_3 + 2d_4 + \cdots = 2 + \sum_{i=3}^{n} d_i$$

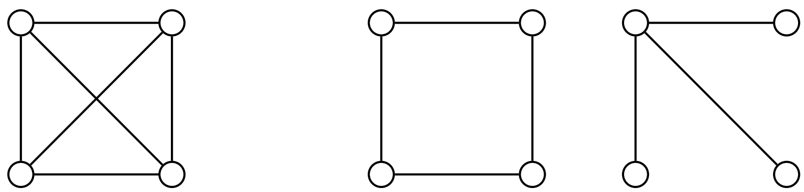
This equation stems from our calculations in corollary 1.

8.2 Spanning

Definition

A subgraph H of graph G is **spanning** if $V(H) = V(G)$.
A **spanning tree** T of a graph G is a tree T that is a subgraph of G such that $V(G) = V(T)$.

Example 8.2.1. *Consider K_4. Observe that the two adjacent graphs are spanning trees of K_4*



Generally, there are a lot of spanning trees for every graph.
A natural question that arises from studying spanning trees is *which graphs have a spanning tree?* Well, for such a graph G, we know that the spanning tree T will contain all of the same vertices as G, and since T is a tree, it must be connected, hence we know that G will be connected as well. This observation raises the question

Does every connected graph have a spanning tree?

The answer is yes! We will prove it in the following theorem.

Theorem 8.1.1

Graph G is connected if and only if G has a spanning tree

Proof of Theorem 8.1.1:

- Proof of $\Longleftarrow$: *If G has a spanning tree, then G is connected*
  Let T be a spanning tree of G, and $x, y \in V(G)$. Since T is spanning, $x, y \in V(T)$. Since T is connected (by definition), there exists a path P from x to y in T. Since T is a subgraph of G, there exists a path P from x to y in G, which implies that G is connected as desired.

- Proof of $\Longrightarrow$: *If G is connected, then G has a spanning tree*
  Let H be a connected spanning subgraph of G and subject to that, $|E(H)|$ is minimized (i.e., find such a connected subgraph H in G that contains the least number of edges). Note that H exists since G is a connected spanning subgraph of itself.

From here, I claim that H is a tree. We already showed H is connected, so all we have to prove is that H doesn't contain a cycle (we'll do so by contradiction):
Suppose not. By our assumption, H is connected, so if H is not a tree, then H contains a cycle. Let C be a cycle of H and let $e \in E(C)$. Let $H' = H - e$. Note that H' is spanning because $V(H') = V(H) = V(G)$. We claim that H' is connected. To see this, let $x, y \in V(H')$. But then $x, y \in V(H)$. Since H is connected, then there exists a path P from x to y in H.
Now, if $e \notin E(P)$, then P is a path in H' as desired.
If $e \in E(P)$, let $W = P - e + (E(C) - e)$. Then W is a walk in H' from x to y. So by theorem, there exists a path in H' from x to y, thus H' is connected as claimed. BUT then, $|E(H')| = |E(H)| - 1$, which contradicts the minimality of H. This proves the claim of H being a tree and hence a spanning tree as desired.

Corollary of Theorem 8.1.1

Let G be a connected graph. $|E(G)| = |V(G)| - 1$ *if and only if G is a tree*

Proof of Corollary:

- Proof of $\Longleftarrow$: *if G is a tree, then* $|E(G)| = |V(G)| - 1$
  We proved this in the last lecture ;)

- Proof of $\Longrightarrow$: *if* $|E(G)| = |V(G)| - 1$*, then G is a tree*
  Suppose $|E(G)| = |V(G)| - 1$. By theorem, if G is connected, G has a spanning tree T. Since T is a tree, $|E(T)| = |V(T)| - 1$. Since T is spanning, $V(T) = V(G)$. Thus, $|E(T)| = |V(G)| - 1$. Yet by supposition, $|E(G)| = |V(G)| - 1$. Thus, since T is a subgraph of G, T and G have the same edges! But since T is spanning, T and G also have the same vertices! Thus, $T = G$, hence G is a tree as desired.

8.2.1 Algorithm to Decide if a Graph is Connected

Let G be a graph

1. Pick $v \in V(G)$. Let T = v
2. While $\delta(V(T)) \neq \emptyset$; let $e \in \delta(V(T))$. Set $T = T + e$
3. If $V(T) = V(G)$, then T is a spanning tree. If not, then let $X = V(T)$ and realize that $\delta(X)$ is empty!