

# MATH 239 — LECTURE 7

Bartosz Antczak

Instructor: Luke Postle

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## Review of last lecture

Let  $X$  be a subset of  $V(G)$ . The **cut** induced by  $X$ , denoted by  $\delta(X)$ , is the set of edges with one end in  $X$  and the other not in  $X$ .

We also studied a **theorem** relating connectivity with  $\delta(X)$ :

*Graph  $G$  is disconnected if and only if there exists a proper, non-empty subset of  $X$  of  $V(G)$  such that  $\delta(X)$  is empty*

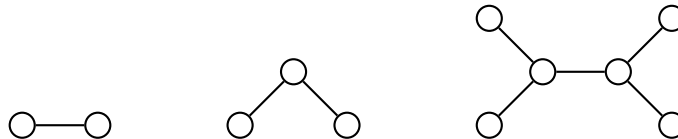
We also mentioned three concepts of algorithm analysis:

- $P$ : polynomial time to answer/decide
- $NP$ : polynomial time to convince there exists a solution
- $Co-NP$ : polynomial time to convince there does not exist a solution

We also learned about **trees** and **forests**:

- Tree: a connected graph containing no cycle
- Forest: a graph (can be either connected or disconnected) containing no cycle

**Example 7.0.1.** *Some examples of trees*



The question we'll focus on today is *how can I convince you that a tree doesn't contain a cycle?*

## 7.1 More on Trees and Forests

### 7.1.1 Leaves

A **leaf** is a vertex of degree one.

**Theorem 7.1.1:**

*If  $T$  is a tree with at least two vertices, then  $T$  has a leaf*

**Proof of Theorem 7.1.1:** to prove this theorem, we'll instead prove the following (it will end up proving that  $T$  will have at least *two* leaves):

*If  $G$  has a minimum degree of at least 2, then  $G$  contains a cycle*

**Proof:** Let  $P = v_0v_1 \cdots v_k$  be a longest path in  $G$ . Since  $G$  has minimum degree of at least 2, then  $v_0$  has at least two neighbours (one of which is  $v_1$ ). Thus, there exists a neighbour  $u$  of  $v_0$  distinct from  $v_1$ . If  $u \in V(P)$ , say  $u = v_i$ , then  $C = v_0v_1, \dots, v_iv_0$  is a cycle of  $G$ , as desired. Now suppose  $u \notin V(P)$ , but then  $P' = uv_0v_1 \cdots v_k$  is a path that is longer than  $P$ , a contradiction. Now, let's prove

*If  $T$  is a tree with at least two vertices, then  $T$  has at least two leaves*

**Proof:** Let  $P = v_0v_1 \cdots v_k$  be a longest path. Note that  $k \geq 1$  because  $T$  is connected with at least two vertices. Now  $v_0$  does not have a neighbour  $u \neq v_1$ , for if  $u \notin V(P)$ , then  $P' = uv_0v_1 \cdots v_k$  is a longer path, a contradiction.

If  $u \in V(P)$ , say  $u = v_i$ ,  $C = v_0v_1, \dots, v_iv_0$  is a cycle, contradicting that it's a tree.

So  $v_0$  has degree one (i.e., is a leaf). But by symmetry,  $v_k$  is also a leaf, and since  $k \geq 1$ ,  $v_0 \neq v_k$ , so  $T$  has at least two leaves, as desired.

**Theorem 7.1.2:**

*If  $T$  is a tree and  $v$  is a leaf of  $T$ , then  $T - v$  is a tree*

**Proof of Theorem 7.1.2:**  $T - v$  is a forest, because  $T - v$  is a subgraph of  $T$ , and every subgraph of a forest is a forest ( $T$  is a tree, but by definition, it's a forest too).

Now we want to claim  $T - v$  is connected and hence a tree. To see this, let  $x, y \in V(T) - v$ . Since  $T$  is connected, there exists a path  $P$  from  $x$  to  $y$  in  $T$ . Note that  $v$  is not in  $P$  because  $v$  has degree one and is not an end of  $P$ . But then  $P$  is a path from  $x$  to  $y$  in  $T - v$ , so  $T - v$  is connected, as desired.

**Corollary**

*$T$  is a tree if and only if there exists a sequence of vertices  $v_1, v_2, \dots, v_{n-1}$  such that if  $T_i = T - \{v_1, v_2, \dots, v_{n-1}\}$ , then  $T_i$  is a tree*

This shows that deciding if  $G$  is a tree is in  $NP$ .

**Algorithm for Deciding if  $G$  is a Tree**

From the proven theorems shown above, we can create an algorithm for determining if  $G$  is a tree:

- If  $|V(G)| = 1$ , then yes!
- If  $|V(G)| \geq 2$ , determine if  $G$  has a leaf:
  - If yes, delete the leaf  $v$  and recurse on  $G - v$
  - If no, then no!

This means that deciding if  $G$  is a tree is in polynomial time.

**Theorem 7.1.3:**

*If  $T$  is a tree, then  $|E(G)| = |V(G)| - 1$*

**Proof of Theorem 7.1.3:** Proceed by induction on  $|V(T)|$ :

*Base case:* If  $|V(T)| = 1$ , then  $|E(G)| = 0$ , as desired.

*Inductive case:* If  $|V(G)| \geq 2$ , then  $T$  has a leaf  $v$ , by our earlier theorem. by previous theorem,  $T - v$  is a tree. By induction,  $|E(T-v)| = |V(T-v)| - 1$ . However,  $|V(T)| = |V(T-v)| + 1$ , and  $|E(T)| = |E(T-v)| + 1$ , since  $v$  has degree one. Thus,  $|E(G)| = |V(G)| - 1$ , as desired.