MATH 239 — Lecture 6

Bartosz Antczak *Instructor: Luke Postle* *January 16, 2017*

Review of last lecture

Path-connectedness (whether two vertices are connected by a path) is an equivalence relation. The equivalence classes are called the components of G. A graph is connected if and only if it has one component.

6.1 Algorithmic Questions for Connectedness

When we study algorithmic questions, we often ask ourselves about how efficient they are. Another topic to focus on is how efficiently can we *confirm* that the algorithm is correct, from this stems the idea of P and NP (and also co-NP):

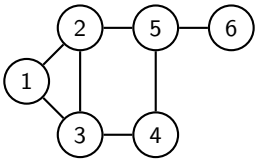
- P (polynomial time) — an algorithm that runs in time polynomial of its input
- NP (non-deterministic polynomial time) — checking whether an algorithm is correct (i.e., if the output of a certain algorithm is correct, I can convince you in polynomial time that it's correct, or equivalently if I'm allowed to guess as much as I want for free, it's in P)
- Co-NP — checking whether the output of an algorithm is incorrect (i.e., I can convince you that the answer is incorrect, thus there being no solution, in polynomial time)

Since most students in this class are CS majors, we'll focus on this approach to analysing problems in graph theory.

Definition — "Cut"

Let $X \subseteq V(G)$. The *cut* induced by X is the set of edges with one end in X and the other not in X. This set is denoted $\delta(X)$.

Example 6.1.1. *Consider the following graph. If we let $X = \{1, 2, 3\}$, then $\delta(X) = \{25, 34\}$, which are the two edges that link the rest of the graph with X.*



Theorem 1

G is disconnected if and only if there exists a proper, non-empty set X of $V(G)$ such that the cut induced by X is empty.

Proof: To prove this, we must show that this statement holds in both "directions" (since it's an iff statement). The proof will conclude on the next page.

(proof of theorem 1 continued)

- Proof in the $\implies$ direction: *if G is disconnected, there there exists such a cut.*
  Since G is disconnected, G has at least two components. Let X be te set of vertices of one component of G. Then $\delta(X)$ is empty since there are no edges between different components.

- Proof in the $\impliedby$ direction: *if there exists a cut, then G is disconnected.*
  Suppose not (we're going by contradiction). Let u be a vertex in X and v not in X (u exists since X is non-empty, and v exists since X is proper). Since we suppose G *is* connected, there exists a path P from u to v, but then there exists an edge $e \in P$ with one end in X and the other not in X, contradicting that $\delta(X)$ is empty.

Proving Whether a Graph is Connected or Not

To prove a graph G is

- *disconnected*: use the previous theorem and exhibit such a set X that satisfies the requirements for a disconnected graph

- *connected*: show that there exists a path between every pair of vertices (but actually only need a path from one vertex to rest of the vertices, or in other words a "common central point" in the graph; refer to theorem 2)

Theorem 2

If $v \in V(G)$ and there is a path from v to every vertex of G, then G is connected.

Proof: Let $u, w \in V(G)$. By our assumption, there exists a path from v to u and also from v to w. We can connect the two paths to form a link from w to v, and then from v to u. If this link is a path, we have proven that there exists a path between any two pairs of vectors in G, thus showing that G is connected. If the link is a walk, using a corollary proven in lecture 4, any walk can be converted to a path, and thus we arrive at the same conclusion.

6.2 Trees and Forests

6.2.1 Definition:

A graph is a **tree** if it contains no cycles. A **forest** is a group of trees. Every tree is a forest (converse not true), and every component of a forest is a tree.

Proposition

If H is a subgraph of a forest G, then H is a forest.

Proof: Suppose not. Then H contains a cycle, call it C. But if G contains C, then G is not a forest — a contradiction.

Final note: it's not true that every subgraph of every tree is itself a tree.