MATH 239 — LECTURE 4

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Recall from last lecture — Handshaking Theorem

In a graph $G = (V, E)$

$$\sum_{v \in V} \deg(v) = 2|E|$$

A corollary that stems from this states that the average degree of a vertex in $G = (V, E)$ is $\frac{2|E|}{|V|}$

Definition: k-regular

A graph is **k-regular** if every vertex in it has degree k. Some examples include

- the Petersen graph (3-regular)
- the complete graph (K_n is $(n-1)$-regular)
- cycle graphs (C_n is 2-regular for all n)

By the handshaking theorem, if $G = (V, E)$ is a k-regular graph, then

$$|E| = \sum_{v \in V} \deg(v) = \frac{1}{2}k|V|$$

For instance, using this equation, the complete graph K_n is $(n-1)$-regular and has n vertices, so

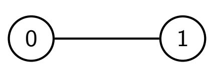
$$|E(K_n)| = \frac{1}{2}n(n-1) = \binom{n}{2}$$

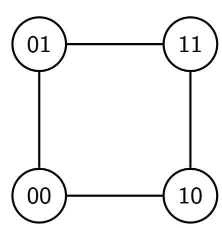
Definition: n-cube

The **n-cube**, Q_n is defined by

- $V(Q_n) = \{\text{binary string of length } n\}$; $|V(Q_n)| = 2^n$
- $E(Q_n) = \{\text{pairs of strings that differ in exactly one position}\}$; $|E(Q_n)| = n \cdot 2^{n-1}$

To better visualize this, shown are a 1-cube and 2-cube graph:



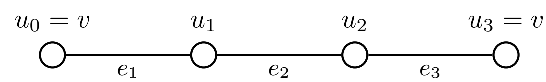


4.1 Paths and Walks

Recall the path on n vertices P_n has vertex set $\{1, \cdots, n\}$ and edge set $\{\{i, i+1\} \,|\, 1 \leq i < n\}$.
A **path** from u to v in a graph is a sequence $u_0, e_1, u_1, e_2, \cdots, u_{k-1}, e_k, v_k$ where

- $u_0 = u$ and $u_k = v$
- For $1 \leq i \leq k$, u_i are distinct vertices of G and e_i is an edge of G from u_{i-1} to u_i

Shown is an example of a simple path (note that $u_i \neq u_j$ for all $i \neq j$):



A **walk** from u to v in G is defined in the same way as a path, except we drop the requirement that the vertices are distinct.
A good distinction between a walk and a path is that if there exists a walk from u to v and from v to w, then there exists one from u to w; but the same isn't true for a path, because the path from v to w may intersect with some vertices from the path from u to v, causing not all vertices being distinct, ergo not a path (*however*, we can convert it to a path — this is proven in the corollary below).

4.1.1 Proposition about Paths

If there is a walk from u to v in G, then there is a path from u to v.

Proof: Let $u = u_0, e_1, u_1, e_2, \cdots, e_k, u_k$ be the shortest walk from u to v. If $u_0, u_1, \cdots, u_k$ are distinct, then the walk is a path and there is nothing to prove. Otherwise, $u_i = u_j$ for some $i < j$. But then $u = u_0, e_1, u_1, e_2, \cdots, u_i = u_j, e_{j+1}, u_{j+1}, \cdots, e_k, u_k$ is a shorter walk (a.k.a. has fewer edges) from u to v, which is a contradiction. Thus, a walk can always be converted to a path.

Corollary

If u, v, w are vertices of a graph G and there is a path from u to v and a path from v to w, then there is a path from u to v.

Proof: Combine both paths from u to v and v to w. If this combination is also a path, then we're done. If it's a walk, then the previous proposition shows that all walks can be converted to paths. Thus, there is a path from u to v.