MATH 239 — Lecture 33

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33.1 Repeated Roots in Recurrence Relations

Last lecture we focused on recurrence relations, but their roots were distinct. Say we have $A(x) = \frac{1}{(1-2x)^2}$. Can we find an explicit expression for a_n? Yes we can! We simply solve

$$[x^n]\frac{1}{(1-2x)^2}$$

We solve this using our handy dandy negative binomial: $(1-y)^{-m} = \sum_{n\geq 0}\binom{n+m-1}{m-1}y^n$

$$\begin{aligned}[x^n](1-2x)^{-2} &= [x^n]\sum_{n\geq 0}\binom{n+2-1}{2-1}(2x)^n \\ &= [x^n]\sum_{n\geq 0}\binom{n-1}{1}2^n x^n \\ &= (n+1)2^n \qquad \square\end{aligned}$$

Example 33.1.1. *Solve the following recurrence relation: $a_n = 6a_{n-1} + 9a_{n-2}$, with $a_0 = a_1 = 1$*

Solution to Example 33.1.1

We define $Q(x) = 1 - 6x + 9x^2$. From this, our characteristic polynomial is $x^2 - 6x + 9 = (x-3)^2$.

$$\begin{aligned}a_n &= (An+B)3^n \\ a_0 = 1 &\implies B = 1 \\ a_1 = 1 &\implies A = -\frac{2}{3}\end{aligned}$$

Thus, a_n in closed form is written as

$$a_n = \left(-\frac{2n}{3} + 1\right)3^n$$

33.1.1 Theorem 1

Suppose $(a_n)_{n\geq 0}$ satisfies some recurrence relation $a_n + q_1 a_{n-1} + \cdots + q_k a_{n-k}$, $n \geq k$. If the characteristic polynomial has roots r_i with multiplicity $m_i, q \leq i \leq j$, the general solution is

$$a_n = P_1(n)r_1^n + \cdots + P_j(n)r_j^n$$

where $P_i(n)$ is a polynomial of degree less than m_i

Example 33.1.2. *Find a_n explicitly, where $a_n - 4a_{n-1} + 5a_{n-2} - 2a_{n-3} = 0$ with initial conditions $a_0 = 1, a_1 = 1, a_2 = 2$.*

Solution to Example 33.1.2

- First step: find $Q(x)$

$$Q(x) = 1 - 4x + 5x^2 - 2x^3$$

From this, our characteristic polynomial is:

$$C(x) = x^3 - 4x^2 + 5x - 2 = (x-1)^2(x-2)$$

- Next step: apply theorem 3.2.2. We'll have a solution of the form

$$a_n = (An + B)1^n + (c)2^n \qquad n \geq 0$$

- Final step: refer to initial conditions to solve for A and B:

$$\begin{aligned} a_0 = 1 &\implies 1 = B + c \\ a_1 = 1 &\implies 1 = A + B + 2c \\ a_2 = 2 &\implies 2 = 2A + B + 4c \end{aligned}$$

Solving these equations gives us: $A = -1,\ B = 0,\ C = 1$. Thus

$$a_n = 2^n - n,\ n \geq 0$$

Example 33.1.3. *Find c_n explicitly, where $c_n + 4c_{n-1} - 3c_{n-2} - 18c_{n-3} = 0,\ n \geq 3$ with initial conditions $c_0 = 0, c_1 = 2, c_2 = 13$.*

Solution to Example 33.1.3

Let $C(x)$ be our characteristic polynomial. Then

$$C(x) = x^3 + 4x^2 - 3x - 18 = (x-2)(x+3)^2$$

This polynomial has roots $2,\ -3$ with multiplicity $1,\ 2$ respectively (this means that the root 2 will have only a constant next to it, and the root (-3) will have a linear polynomial next to it. We get from theorem 3.2.2 that c_n will be of the form:

$$c_n = A \cdot 2^n + (B + cn)(-3)^n$$

From our initial conditions, we solve for the constants, which are $A = 1,\ B = -1,\ C = 1$. To conclude, we solve

$$c_n = 2^n + (n-1)(-3)^n,\ n \geq 0$$