

MATH 239 — LECTURE 32

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Recall from last lecture

$$\binom{1/2}{k} = \frac{(-1)^{k-1}(1 \cdot 2 \cdot 3 \cdots (2k-3)(2k-2))}{2^k k! 2^{k-1}(1 \cdot 2 \cdots k-1)} = \frac{(-1)^{k-1}}{k \cdot 2^{2k-1}} \binom{2k-2}{k-1}$$

By Binomial theorem

$$(1+x)^{1/2} = 1 + \sum_{k \geq 1} \frac{(-1)^{k-1}}{k \cdot 2^{2k-1}} \binom{2k-2}{k-1} x^k$$

We can use this to solve

$$(1-4x)^{1/2} = 2x \left(\sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n \right) - 1$$

This content will **not** be on the final.

32.1 Recurrence Relations

32.1.1 Theorem 1

Let

$$A(x) = a_0 + a_1x + a_2x^2 + \cdots$$

$$P(x) = p_0 + p_1x + p_2x^2 + \cdots$$

$$Q(x) = 1 - q_1x - q_2x^2 - \cdots$$

be formal power series. Then

$$Q(x)A(x) = P(x) \iff \forall n \geq 0, a_n = p_n + q_1a_{n-1} + q_2a_{n-2} + \cdots + q_na_0$$

Example 32.1.1. Consider the following formal power series.

$$A(x) = \frac{1+x}{1-5x+6x^2}$$

$$P(x) = 1+x \implies p_0 = 1, p_1 = 1$$

$$Q(x) = 1-5x+6x^2 \implies q_1 = 5, q_2 = 6$$

Using partial fractions, we have

$$A(x) = \frac{1+x}{1-5x+6x^2} = \frac{a}{1-2x} + \frac{b}{1-3x} \implies a = 3, b = 4$$

Finding the n th coefficient yields:

$$[x^n]A(x) = [x^n] \left(\frac{-3}{1-2x} \right) + [x^n] \left(\frac{4}{1-3x} \right) \implies a_n = (-3 \cdot 2^n + 4 \cdot 3^n)$$

For a_n , where do the 2 and 3 come from? They're the roots of

$$P = x^{\deg Q} Q(x^{-1})$$

We define P as the *characteristic polynomial* of A .

32.1.2 Theorem 2

Suppose $(a_n)_{n \geq 0}$ satisfies the recurrence relation $a_n + q_1 a_{n-1} + \dots + q_k a_{n-k} = 0$, ($n \geq k$).

If the characteristic polynomial of this has root r_i for $i = 1, \dots, j$, then the general solution is

$$a_n = c_1 r_1^n + c_2 r_2^n + \dots + c_j r_j^n$$

The previous theorem holds for **no repeated roots**.

32.1.3 Theorem 3

Let $A(x) = \sum_{n \geq 0} a_n x^n$ where a_n satisfies $a_n + q_1 a_{n-1} + \dots + q_k a_{n-k} = 0$ ($n \geq k$).

If $g(x) = 1 + q_1 x + q_2 x^2 + \dots + q_k x^k$, then there exists a polynomial f of degree $< k$ such that

$$A(x) = \frac{f(x)}{g(x)}$$

Proof of Theorem 3

By the rule for finding coefficients of products of power series,

$$a_n + q_1 a_{n-1} + \dots + q_k a_{n-k} = [x^n](1 + a_1 x + a_2 x^2 + \dots + a_k x^k)A(x)$$

So $[x^n]g(x)A(x) = 0$ if $n \geq k$, and so

$$g(x)A(x) = f(x), \deg(f) < k$$

Example 32.1.2.

Recall the Fibonacci Numbers: $a_n = a_{n-1} + a_{n-2} \implies a_n - a_{n-1} - a_{n-2} = 0$

So $g(x) = 1 - x - x^2$. From this, our characteristic polynomial is $x^{\deg(g)}g(x^{-1}) = x^2 - x - 1$. From this, our roots to the characteristic polynomial are:

$$r_1, r_2 = \frac{1 \pm \sqrt{5}}{2}$$

That tells us that $a_n = c_1 r_1^n + c_2 r_2^n$. To find c_1 and c_2 , use initial conditions (i.e., a_0 and a_1 , both 1)

- For $n = 0$, $c_1 + c_2 = a_0 = 1$

- For $n = 1$, $c_1 \left(\frac{1 + \sqrt{5}}{2} \right) + c_2 \left(\frac{1 - \sqrt{5}}{2} \right) = 1$

From this,

$$c_1 = \frac{1 + \sqrt{5}}{2\sqrt{5}}, \quad c_2 = \frac{1 - \sqrt{5}}{2\sqrt{5}}$$