MATH 239 — Lecture 30

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30.1 More on Binary Strings

Let's return to our problem we outlined last lecture:

How many binary strings are there of length n with no 3 consecutive ones?

To better tackle this problem, we create a definition.

Definition — substring

We say b is a **substring** of a if $\exists\, c, d$ such that $a = cbd$.
From this definition, we can restructure our question as such:

How many strings of length n have no 111 substring?

If we let D be the binary strings with no 111 substring, our goal is to find $\Phi_D(x)$.

30.1.1 Solving our Problem

Let B be the set of all binary strings. We ask ourselves, what is an unambiguous expression for B?

- The trivial solution is $\{0,1\}^*$ (here, we decompose after every 0 or 1)
- Another approach is to only decompose after every 0. We'll stick with this one

Under this decomposition, one piece can look like:

$$0, 10, 110, 1110, \cdots$$

In other words, some number of 1s followed by 0. A more compact way to write this piece is $P = \{1\}^*\{0\}$. However, we can have *any* number of pieces:

$$B = (\varepsilon \cup P \cup P^2 \cup P^3 \cup \cdots)\{1\}^* = P^*\{1\}^* = (\{1\}^*\{0\})^*\{1\}^*$$

(Here, we concatenated $\{1\}^*$ at the end to ensure we have matched to every binary string). We state that B is unambiguous. Why? Because it came from a decomposition rule that *uniquely* decomposes every string.
Now for D, what is the equivalent expression? Under our decomposition, what can one piece in D look like?

- Beginning/middle piece: $0, 10, 110$
- End piece: $\varepsilon, 1, 11$

For D, our expression is: $D = (\{0, 10, 110\})^*\{\varepsilon, 1, 11\}$ and this is unambiguous, since it is just a restriction of an unambiguous expression for B.

Finishing Off

So,

$$\begin{aligned}
\Phi_D(x) &= \Phi_{\{0,10,110\}^*} \cdot \Phi_{\{\varepsilon,1,11\}}(x) && \text{(By Product Lemma)}\\
&= \frac{1}{1-\Phi_{\{0,10,110\}^*}} \cdot (1+x+x^2) && \text{(By our Star Lemma)}\\
&= \frac{1+x+x^2}{1-(x+x^2+x^3)} && \square
\end{aligned}$$

From here, we can solve for the number of binary strings of length n by solving $[x^n]\Phi_D(x)$.

30.1.2 Other Decomposition Rules

- **Decompose after every 1** (i.e., symmetric version to decomposing after every 0): $B = (\{0\}^*\{1\})^*\{0\}^*$ is useful for solving problems regarding the numbers of 0s in a string

We have a more general decomposition rule. Before we state it, let's create a definition:

Definition — Block

A **block** of a binary string is a maximal substring all having the same digit (i.e., all 0s or all 1s). For example, in 00111011, our blocks are: $00, 111, 0, 11$.

- **Block Decomposition Rule**:
  Decompose after every block of 0s:
  - beginning piece: $\{0\}^*$
  - middle piece: $\{1\}^*\{1\}\{0\}^*\{0\}$
  - end piece: $\{1\}^*$

  Then, $B = \text{beginning} \cdot (\text{middle})^* \cdot \text{end} = \{0\}^*(\{1\}^*\{1\}\{0\}^*\{0\})^*\{1\}^*$

30.1.3 Problems from Course Notes

2.7.1. *Find the generating series for binary strings with no block of exactly length 2.*

Solution

To solve this, let us restrict the block rule to the condition: "no block of length exactly two".

- Beginning piece: $\{\varepsilon, 0, 000, 0000, \cdots\} = \varepsilon \cup \{0\} \cup \{000\}\{0\}^*$
- Middle piece: $(\{1\} \cup \{111\}\{1\}^*)(\{0\} \cup \{000\}\{0\}^*)$
- End piece: $\varepsilon \cup \{1\} \cup \{111\}\{1\}^*$

Then, our expression is:

$$(\varepsilon \cup \{0\} \cup \{000\}\{0\}^*)((\{1\} \cup \{111\}\{1\}^*)(\{0\} \cup \{000\}\{0\}^*))^*(\varepsilon \cup \{1\} \cup \{111\}\{1\}^*)$$

To find the geometric series for each piece:

$$\Phi_{begin}(x) = 1 + x + x^3 \cdot \frac{1}{1-x}$$
$$\Phi_{mid}(x) = x + x^3 \cdot \left(\frac{1}{1-x}\right)\left(x + x^3 \cdot \frac{1}{1-x}\right)$$
$$\Phi_{end}(x) = 1 + x + x^3 \cdot \frac{1}{1-x}$$

So our generating series is:

$$= \left(1 + x + \frac{x^3}{1-x}\right)\left(\left(x + \frac{x^3}{1-x}\right)^2\right)^* \left(1 + x + \frac{x^3}{1-x}\right)$$
$$= \left(1 + x + \frac{x^3}{1-x}\right)\left(\frac{1}{1-(x+\frac{x^3}{1-x})^2}\right)\left(1 + x + \frac{x^3}{1-x}\right)$$
$$= \frac{1 - x^2 + x^3}{1 - 2x + x^2 - x^3} \qquad \square$$