MATH 239 — Lecture 29

Bartosz Antczak *Instructor: Luke Postle* *March 20, 2017*

29.1 Binary Strings

Recall the Definition

A **binary string** is a sequence, where each digit is a zero or one. The length of the string is equal to the number of digits. There are 2^n binary string of length n, for all n (which includes the empty string ε. Here, $n = 0$, so there is $2^0 = 1$ possible string, which is the empty string).

Question 1

How many binary strings are there of length n with no 3 consecutive ones?

Today we'll solve this problem (if you took STAT 230, you'll know how to calculate this, but chances are you don't know how to calculate it *combinatorially*.

29.1.1 Operations on Binary Strings

Union

Example 29.1.1. *Union and of binary strings:* $A = \{0, 01\}$, $B = \{0, 10\} \implies A \cup B = \{0, 01, 10\}$

Define the weight of binary numbers to be their length. If A is a set of binary strings, $\Phi_A(x)$ will be its generating series.

Proposition 1

If $S = A \sqcup B$, then $\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$

Product

Definition 1: string concatenation

If a, b are two binary stings, then the **concatenation** of a and b, written ab, is the sequence obtained from a by appending b.

Example 29.1.2. *Concatenating two binary strings:* $a = 101$, $b = 00 \implies ab = 10100$

Definition 2: set concatenation

If A, B are two *sets* of binary strings, then the **concatenation** of A and B, denoted AB, is the set of *unique* strings s such that $\exists\, a \in A, b \in B$ with $s = ab$ (i.e., all strings which can be formed as the concatenation of an element of A with an element of B).

Example 29.1.3. $A = \{0, 01\}, B = \{0, 10\} \implies AB = \{00, 010, 0110\}$

Observe that in the above example, 101 can be made twice, but by our definition, AB is not a multi set, which means that every element AB contains must be *unique*. Note that the product lemma doesn't apply to this example: $\Phi_{AB}(x) \neq \Phi_A(x)\Phi_B(x)$

Definition 3: ambiguity

We say AB is **unambiguous** of $\nexists\, s \in AB$ and $a_1, a_2 \in A, b_1, b_2 \in B$ such that $s = a_1b_1 = a_2b_2$. In other words, no string in AB can be made in 2 or more ways).
Equivalently, $\forall\, s \in AB$ is the *unique* concatenation of an element of A and an element of B.
Obviously, we say AB is **ambiguous** if it's not unambiguous.

Example 29.1.4. $A = \{0, 1, 11\}$, $B = \{10, 01\} \implies AB = \{010, 001, 110, 101, 1110, 1101\}$ *is unambiguous since there exist no multiples in AB, ergo* $|AB| = |A||B|$

Proposition

$$|AB| = |A||B| \iff AB \textit{ is unambiguous}$$

Example 29.1.5. $A = \{0, 1, 11\}$. *Is* A^2 *unambiguous?* $A^2 = \{00, 01, 011, 10, 11, 111, 110, 1111\}$. *No it isn't, 111 can be made in two ways.*

Lemma 1

$$\textit{If } AB \textit{ is unambiguous, then } \Phi_{AB}(x) = \Phi_A(x)\Phi_B(x)$$

Similarly, if we have many concatenations (e.g., $S = ABC$), then we say S is unambiguous if every string in S has a *unique* way to be made.

General Lemma 1

$$\textit{If } S = A_1 \cdot A_2 \cdots A_k \textit{ is unambiguous, then } \Phi_S(x) = \prod_{i=1}^{k} \Phi_{A_i}(x)$$

Final operation: star

$A*$ is the set of all strings which can be made by concatenating elements of A.

$$A^* = \varepsilon \cup A \cup A^2 \cup A^3 \cup \cdots$$

Example 29.1.6. $A = \{0, 1\} \implies A^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, \cdots\}$ *(the set of all binary strings)*

Definition: ambiguity with stars

We say A^* is unambiguous if

- A^k is unambiguous $\forall\, k$
- A^i is disjoint from $A^j, \forall\, i \neq j$, even $A^0 = \varepsilon$

Lemma 2

If A^* is unambiguous, then

$$\Phi_{A^*}(x) = \Phi_{A^0}(x) + \Phi_{A^1}(x) + \Phi_{A^2}(x) + \cdots = \sum_{k=0}^{\infty} \Phi_A(x)^k = \frac{1}{1 - \Phi_A(x)}$$

Note: if $\varepsilon \in A$, then A^* is *ambiguous.*

More generally...

A series of operations is unambiguous if all concatenations are unambiguous and unions are disjoint.

Example 29.1.7. *Applying these operations to find the number of binary strings of length n*

Let $A = \{0, 1\}$, $\Phi_A(x) = 2x$. Then the set of all binary strings is $B = A^*$. This means that:

$$\Phi_B(x) = \frac{1}{1 - \Phi_A(x)} = \frac{1}{1 - 2x} = \sum_{n=0}^{\infty} 2^n x^n$$

Getting back to our "3 consecutive ones" problem

We define the set of all binary strings without three consecutive ones as:

$$B = \{0, 10, 110\}^* \{\varepsilon, 1, 11\}$$

If you believe this to be unambiguous, we have

$$\Phi_B(x) = (x + x^2 + x^3)^*(1 + x + x^2) = \frac{1 + x + x^2}{1 - (x + x^2 + x^3)}$$