MATH 239 — Lecture 26

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Review

Negative binomial series (remember this for the exam):

$$\frac{1}{(1-x)^k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} x^n$$

26.1 The Inverse of Power Series

We state a theorem:

26.1.1 Theorem 1

$$B(x) \text{ has an inverse } \iff b_0 \neq 0$$

Proof of Theorem 1

- **($\Longrightarrow$):**
  If $b_0 = 0$, then $B(x)$ has no inverse $D(x)$ such that $D(x) \cdot B(x) = 1$ $\square$

- **($\Longleftarrow$):**
  Suppose $b_0 \neq 0$. We want to find $D(x)$ such that $D(x) \cdot B(x) = 1$. With $D(x) \cdot B(x) = 1$ we want:

  $$d_0b_0 + (d_0b_1 + d_1b_0)x + (d_0b_2 + d_1b_1d_2b_0)x^2 + \cdots = 1 + 0x + 0x^2 + \cdots$$

  But then the coefficients must be equal, giving rise to an infinite number of equations:

  - Constant: $d_0b_0 = 1$
  - Linear: $d_0b_1 + d_1b_0 = 0$
  - Quadratic: $d_0b_2 + d_1b_1 + d_2b_0 = 0$

  From these equations, the d_i are uniquely determined as follows:

  $$d_0 = \frac{1}{b_0}$$
  $$d_1 = -\frac{b_1}{b_0^2}$$
  $$d_0 = \frac{-d_0b_2 - d_1b_1}{b_0}$$

  Or to generalize

  $$d_0 = \frac{1}{b_0}$$
  $$\forall\, n \geq 1 \quad d_n = \frac{-(d_{n-1}b_1 + d_{n-2}b_2 + \cdots + d_0b_{n-1})}{b_0} \qquad \square$$

26.1.2 Corollary 1

If the inverse of a formal power series exists, the it's unique. Moreover, the inverse of the inverse is itself

Example 26.1.1.

Consider $\frac{1}{1-2x}$ and $\frac{1}{1-x^2}$? Do these expressions have an inverse? They **do**, since $b_0 \neq 0$.

26.1.3 Definition — Composition

The **composition** of $A(x)$ and $B(x)$ is

$$A(B(x)) = a_0 + a_1 B(x) + a_2 B(x)^2 + \cdots$$

Question

When are compositions well-defined? Particularly, considering its sum up to infinity (for instance, $\displaystyle\sum_{i=1}^{\infty} \frac{1}{x^2}$ is well-defined, whereas $\displaystyle\sum_{i=0}^{\infty} (-1)^n$ is not).

26.1.4 Theorem 2

$$A(B(x)) \text{ is well-defined} \iff \text{either } A(x) \text{ is finite } \underline{\text{or}}\ b_0 = 0$$

Proof

- If $A(x)$ is finite, then $A(B(x))$ is the finite sum of a formal power series, and so it's well-defined
- If $b_0 = 0$, then the smallest term in $B(x)^n$ has degree at least n (since the constant is zero), but then

  $$\begin{aligned} [x^n]A(B(x)) &= \sum_{m \geq 0} [x^n] a_m B(x)^m \\ &= \sum_{m=0}^{n} [x^n] a_m B(x)^m \end{aligned}$$

  because the nth coefficient of $B(x)^m$ is zero if $m > n$ (so the rest of the sum is zero, and thus it is a finite sum, ergo it's well-defined). □

Example 26.1.2.

Consider $A(x) = \frac{1}{1-x}$ and $B(x) = 2x$. $A(B(x))$ is well-defined because $b_0 = 0$:

$$\begin{aligned}
\frac{1}{1-2x} = A(B(x)) &= \sum_{n\geq 0} a_n B(x)^n \\
&= \sum_{n\geq 0} 1 \cdot (2x)^n && \text{(Since } A(x) = 1 + x + x^2 + \cdots\text{)} \\
&= \sum_{n\geq 0} 1 \cdot 2^n \cdot x^n \\
&= \sum_{n\geq 0} 2^n \cdot x^n
\end{aligned}$$

We observe that this is the generating series for binary strings with the weight being the length of the string, so:

$$\Phi_B(x) = \sum_{n\geq 0} 2^n x^n = \frac{1}{1-2x}$$

26.2 Sum and Product Lemmas

26.2.1 Sum Lemma

If $B = A_1 \sqcup A_2$. where they all have the same weight function w, then $\Phi_B(x) = \Phi_{A_1}(x) + \Phi_{A_2}(x)$

Proof of Sum Lemma

Let $\Phi_B(x) = \sum_{n\geq 0} b_n x^n$, $\Phi_{A_1}(x) = \sum_{n\geq 0} a_{1,n} x^n$, $\Phi_{A_2}(x) = \sum_{n\geq 0} a_{2,n} x^n$. By definition, b_n is equal to the number of elements of B with weight n. Since B is the disjoint union of A_1 and A_2, then b_n is also equal to the sum of the number of elements in A_1 and A_2 of weight n, which is defined by $a_{1,n} + a_{2,n}$. Ergo

$$\Phi_B(x) = \Phi_{A_1}(x) + \Phi_{A_2}(x)$$

26.2.2 Product Lemma

If $B = A_1 \times A_2$ (the Cartesian product) and $w((a_1, a_2) \in B) = w(a_1) + w(a_2)$, then
$\Phi_B(x) = \Phi_{A_1}(x) \cdot \Phi_{A_2}(x)$