MATH 239 — Lecture 25

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25.1 Generating Series

Definition

The **generating series** of a set A (e.g., binary string, compositions, etc.) equipped with a weight function[1] w (e.g., length, number of parts, etc.) is the formal power series whose coefficient of x^n is the number of elements of A of weight n. This is denoted as

$$\Phi_A(x)$$

Now, $\Phi_A(x) = \sum_{n \geq 0} a_n x^n$, where a_n is the number of elements of A of weight n. Equivalently,

$$\Phi_A(x) = \sum_{a \in A} x^{w(a)}$$

Example 25.1.1.

Let B be the set of binary strings, and w be the length. The generating series is

$$\Phi_B(x) = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

To write out the coefficients, we start with the number of possible strings there are of length 0 (which is 1), then the number of strings of length 1 (which is 2), and then we have 4, and then 8, and so on.

Example 25.1.2.

Let C be a set of compositions, and let the weight sum up to n iff We have

$$\Phi_C(x) = 1 + 1x + 2x^2 + 4x^3 + 8x^4 + \cdots$$

Example 25.1.3.

Let S_m be the subsets of $[m]$ (for some fixed m). Let the weight be the size of the subset. We have

$$\Phi_{S_m}(x) = 1 + mx + \binom{m}{2}x^2 + \binom{m}{3}x^3 + \cdots + x^m \qquad \text{(i.e., up until we reach} \binom{m}{m}\text{)}$$

But there is actually a more compact way to write it:

$$\Phi_{S_m}(x) = \sum_{n=0}^{m} \binom{m}{n} x^n = (1+x)^m \qquad \text{(By binomial theorem)}$$

[1] a weight function must be non-negative and an integer

In this class, we let

$$\mathbb{N} = \{1, 2, 3, \cdots\}$$
$$\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$$
$$\mathbb{N}_k = \{k, k+1, k+2, \cdots\}$$

Example 25.1.4.

What are the generating series for $\mathbb{N}, \mathbb{N}_0, \mathbb{N}_k$ with $w(i) = i$?

$$\Phi_{\mathbb{N}}(x) = x + x^2 + x^3 + \cdots$$
$$\Phi_{\mathbb{N}_0}(x) = 1 + x + x^2 + x^3 + \cdots$$
$$\Phi_{\mathbb{N}_k}(x) = x^k + x^{k+1} + x^{k+2} + \cdots$$

What about $\mathbb{N}$ but with $w(i) = i^2$?

$$\begin{aligned}\Phi_{\mathbb{N}}(x) &= 0 + 1x + 0x^2 + 0x^3 + 1x^4 + \cdots \\ &= x + x^4 + x^9 + x^{16} = \cdots\end{aligned}$$

Example 25.1.5.

Consider the Cartesian product of $A = \{1, 3, 5\} \times \{2, 4, 7\}$. Define the weight function $w((a, b)) = a = b$. We can write the generating series using “brute force”:

$$\Phi_A(x) = x^{1+2} + x^{1+4} + x^{1+7} + x^{3+2} + x^{3+4} + x^{3+7} + x^{5+2} + x^{5+4} + x^{5+7}$$

Observe that we can factor this line:

$$(x^1 + x^3 + x^5)(x^2 + x^4 + x^7)$$

We'll cover this more on Friday.

25.2 Power Series

Definition

If $(a_0, a_1, \cdots)$ is a sequence of rational numbers, then

$$A(x) = \sum_{n \geq 0} a_n x^n$$

is a **formal power series**. We let $[x^n]A(x)$ denote the coefficients of x^n, in other words, a_n.

Example 25.2.1.

Solve $[x^2](1+x)^5$. In other words, find the coefficients of x^2 in the power series $(1+x)^5$.
Here, we want to solve

$$[x^2]\sum_{n=0}^{5}\binom{5}{n}x^n \qquad \text{(By theorem)}$$

when $n = 2$. Thus, $[x^2](1+x)^5 = \binom{5}{2} = 10$.

25.2.1 Operations on Formal Power Series

Let $A(x) = \sum_{n\geq 0} a_n x^n$, $B(x) = \sum_{n\geq 0} b_n x^n$ be formal power series. The operations we perform on them are outlined:

- **Addition:** $A(x) + B(x) = \sum_{n\geq 0}(a_n + b_n)x^n$
- **Subtraction:** $A(x) - B(x) = \sum_{n\geq 0}(a_n - b_n)x^n$
- **Multiply by constant:** $kA(x) = \sum_{n\geq 0}(ka_n)x^n$
- **Multiplication of Power Series:** $A(x)\cdot B(x) = \sum_{n\geq 0}\left(\sum_{i=0}^{n} a_i b_{n-i}\right)x^n$

  From this we see that

  $$[x^n]A(x)\cdot B(x) = \sum_{i=0}^{n} a_i b_{n-i}$$

  Example 25.2.2.

  Consider $A(x) = B(x) = 1 + x + x^2 + \cdots$:

  $$A(x)\cdot B(x) = 1 = 2x + 3x^2 + \cdots + (n+1)x^n$$

  We also see that

  $$[x^n]A(x)\cdot B(x) = \sum_{i=0}^{n} a_i b_{n-i} = \sum_{i=0}^{n} 1\cdot 1 = n+1$$

- **Division:** recall the definitions of the **inverse**.

  Example 25.2.3.

  Let $B(x) = 1 - x$. The inverse, or $\frac{1}{B(x)}$, is equal to $1 + x + x^2 + \cdots$

Note about Formal Power Series

We don't care what the value of x is. *We are only concerned about the coefficients.*