MATH 239 — Lecture 24

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Review of last lecture

A **permutation** of $[n] = \{1, 2, \cdots, n\}$ is a *sequence* (i.e., ordered) of distinct elements of $[n]$. The number of permutations of $[n]$ of length k is

$$\frac{n!}{(n-k)!}$$

A **combination** of $[n]$ is a set (i.e., unordered) of distinct elements of $[n]$. The number of combinations of n of size k is "n choose k", denoted

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Example 24.0.1. *Some combinations*

$$\binom{n}{0} = 1$$
$$\binom{n}{1} = n$$
$$\binom{n}{n-1} = n$$
$$\binom{n}{n} = 1$$

24.1 Identities and Proving Combinatorially

Recall that the binomial theorem is

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Let's prove this combinatorially:

Proof of Binomial Theorem

The coefficient of x^k in the expansion of $(1+x)^n$ is equal to the number of terms which have chosen x k times from the product $(1 + x^n)$ But this is equivalent to choosing a k-element subset of $[n]$, i.e., $\binom{n}{k}$.

24.1.1 Definition — Combinatorial Identity

A **combinatorial identity** is an equation which relates combinatorial objects/numbers.

Example 24.1.1. *Combinatorial identity #1*

$$\binom{n}{k} = \binom{n}{n-k}$$

Combinatorial Proof of identity 1

Let $S_{n,k}$ be the subsets of $[n]$ of size k. Recall that (by definition):

$$|S_{n,k}| = \binom{n}{k}$$

Let $S_{n,n-k}$ be the subsets of $[n]$ of size k. So,

$$|S_{n,n-k}| = \binom{n}{n-k}$$

Now all we need to do is show that these two sets are of equal size. We do this using *bijection*.
We claim that $S_{n,k}$ is in bijection with $S_{n,n-k}$ (here, S is a subset of $[n]$ of size k):

$$f : S_{n,k} \to S_{n,n-k} : f(S) = [n] - S$$

Now let's consider the inverse (here, T is the subset of $[n]$ of size $n-k$):

$$f^{-1} : S_{n,n-k} \to S_{n,k} : f^{-1}(T) = [n] - T$$

Thus, $|S_{n,k}| = |S_{n,n-k}|$, ergo

$$\binom{n}{k} = \binom{n}{n-k}$$

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Example 24.1.2. *Combinatorial identity #2*

$$\binom{n}{k} = \binom{n-1}{k-1} = \binom{n-1}{k}$$

Combinatorial Proof of identity 2

Let $S_{n,k}$ denote the subsets of $[n]$ of size k.
Let A_1 be the subsets of $[n]$ of size k that contain n (i.e., the last element in $[n]$). Let A_2 be the subsets of $[n]$ of size k that contain n. Thus,

$$S_{n,k} = A_1 \sqcup A_2 \qquad \text{(The disjoint union)}$$

Hence

$$|S_{n,k}| = |A_1| + |A_2| = \binom{n}{k}$$

What is the size of A_1? We claim that A_1 is in bijection with $S_{n-1,k-1}$. Let's write the bijection (Here, $S \in A_1$):

$$f : A_1 \to S_{n-1,k-1} : f(S) = S - \{n\}$$

The inverse is (Here, $T \in S_{n-1,k-1}$):

$$f^{-1} : S_{n-1,k-1} \to A_1 : f^{-1}(T) = T \cup \{n\}$$

Thus, $|A_1| = |S_{n-1,k-1}|$
What is the size of A_2 We claim that A_1 is in bijection with $S_{n-1,k}$. Let's write the bijection:

$$f : A_2 \to S_{n-1,k} : f(S) = S$$

The inverse is:

$$f : S_{n-1,k} \to A_2 : f(T) = T$$

Thus, $|A_2| = |S_{n-1,k}|$.
So,

$$\binom{n}{k} = |S_{n,k}| = |A_1| + |A_2| = \binom{n-1}{k} + \binom{n-1}{k}$$

Final note about combinations

- **Q:** how many subsets of $[n]$ are there of size k?

$$\binom{n}{k}$$

- **Q:** how many binary strings are there of length n with k 1's?

$$\binom{n}{k}$$

- **Q:** how many binary strings are there of length n with k 0's (or $n-k$ 1's)?

$$\binom{n}{k} = \binom{n}{n-k}$$

- **Q:** how many compositions are there of n with k parts? (we're to give a different bijection of this on the homework)

$$\binom{n-1}{k-1}$$

Now, let's stop counting combinatorially and actually *count algebraically*.

24.2 An Algebraic View of Counting

What if we wanted to find the number of a certain type of object (e.g., the number of binary string of size $n > 5$ that don't contain the substring "101").
What if we think of that type we're looking far as an *unknown variable*. Usually we don't want to count one type (e.g., binary strings), but many types (e.g., binary strings of length n).
Let's **encode** these infinitely many variables as coefficients of an infinite polynomial. We'll focus on this topic for the next two weeks.