MATH 239 — Lecture 23

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Review of Last Lecture

A **composition** of n is an ordered sequence of positive integers whose sum in n. There is one composition of 0, the empty composition. We discussed a theorem:

If $n = 1$*, then there are* 2^{n-1} *compositions of* n

There is a bijective proof of it. Here is a different proof:

23.1 Alternate Proof of Composition Theorem

It's a recursive proof. If we can show that the number of compositions doubles from $n-1$ to n, we will have proven it:

- **Base case:** $n = 1$, there is $1 = 2^{1-1}$ compositions of n, as desired
- **Inductive Case:** let $S_{n,1}$ be the composition of n whose first part is 1.
  Let $S_{n,\geq 2}$ be the set of compositions whose first part is ≥ 2. If we let S_n be the set of compositions of n, then observe that
  $$S_n = S_{n,1} \sqcup S_{n,\geq 2}$$
  because since $n \geq 1$, there is a first part and that part is either 1 or ≥ 2. Thus $|S_n| = |S_{n,1}| + |S_{n,\geq 2}|$.
  For instance

n	$S_{n,1}$	$S_{n,\geq 2}$	S_{n-1}
2	1+1	2	1
3	1+2 1+1+1	3, 2+1	2, 1+1
4	1+3, 1+2+1, 1+1+2, 1+1+1+1	4, 3+1, 2+1+1, 2+2	3, 2+1, 1+1+1, 1+2

  We claim that $S_{n,1}$ is in bijection with S_{n-1}:
  $$f : S_{n,1} \to S_{n-1} \cdot f(1 + a_2 + \cdots + a_k) = a_2 + a_3 + \cdots + a_k$$
  and note $a_2 + \cdots + a_k \in S_{n-1}$ because it's the ordered sequence of positive integers whose sum is $n-1$.
  The inverse is:
  $$f^{-1} : S_{n-1} \to S_{n,1} \cdot f^{-1}(1 + a_2 + \cdots + a_k) = 1 + a_1 + a_2 + \cdots + a_k$$
  Thus, $|S_{n,1}| = |S_{n-1}| = 2^{n-2}$ by induction.
  We claim $S_{n,\geq 2}$ is also a bijection with S_{n-1}:
  $$f : S_{n,\geq 2} \to S_{n-1} : f(a_1 + \cdots + a_k) = (a_1 - 1) + a_2 + \cdots + a_k$$

Note $(a_1 - 1) + \cdots + a_k \in S_{n-1}$ because it sums to $n-1$ and it's an ordered sequence of positive integers since $a_1 \geq 2$, so $a_1 - 1 \geq 1$. The inverse is:

$$f^{-1} : S_{n-1} \to S_{n,\geq 2} : f^{-1}(a_1 + \cdots + a_k) = (a_1 + 1) + a_2 + \cdots + a_k$$

and it's in $S_{n,\geq 2}$ because $a_1 + 1 \geq 2$ since $a_1 \geq 1$.

- **Conclusion:** thus, $|S_{n,\geq 2}| = |S_{n-1}| = 2^{n-2}$, by induction. So

$$|S_n| = |S_{n,1}| + |S_{n,\geq 2}| = 2^{n-2} + 2^{n-2} = 2^{n-1} \qquad \square$$

This proof shows that compositions can be built recursively by:

- adding a first part of size 1
- adding 1 to the first part

Thus, every composition of n ($n \geq 1$) can be obtained uniquely from a sequence of new part/ add 1's whose length is $n-1$ (i.e., these can be thought of as binary strings of length $n-1$).
You can use this to find the bijection from last time, and indeed there is a nice interpretation of the bijection::

$$\sum_{i=1}^{n} 1 = 1 + 1 + \cdots + 1 \qquad (n\,\text{times})$$

For each integer 1, we can either add a new part (leave the 1) to it, or add $1+1$ together. This means that there are $n-1$ choices. For example, let $n = 4$:

$$1 + 1 + 1 + 1$$

If, starting from the leftmost one, we apply the choices: new part, add together, new part, we get:

$$1 + 2 + 1$$

23.2 Permutations and Combinations

A **permutation** of $[n] = \{1, 2, \cdots, n\}$ is an ordered sequence of distinct elements of $[n]$. The length of a permutation is the length of the sequence.

General Question

How many permutations are there of $[n]$ of length k?

Observe that for a set $[n]$, there are:

- n choices for the 1st element
- $\cdots$
- $n-k+1$ choices for the kth element

When $k = n$, the length is equal to $n \times (n-1) \times \cdots \times 1 = n!$ A **combination** of $[n]$ is an unordered sequence of distinct elements of $[n]$. It's size is the number of elements in the sequence.

General Question 2

How many combinations of $[n]$ of size k are there?

We make a proposition:

23.2.1 Proposition 1

The number of permutations of $[n]$ of length k is equal to $k!$ multiplied by the number of combinations of size k

Proof of Proposition 1

A permutation of n of length k is just an ordering of a combination of n of size k. Since the combination has size k, there are $k!$ orderings. $\square$

So the number of combinations of $[n]$ of size k is:

$$\frac{n \times (n-1) \times \cdots \times (n-k+1)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Set versus Sequence

A permutation of length k is a particular ordering of a combination of length k. From the combination, there are $k!$ possible permutations.

A permutation can be though of as a particular sequence (or ordered list) of a set of length k (unordered elements), or what we formally call in MATH 239, a *combination*.