MATH 239 — Lecture 20

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Review of Last Lecture

We covered *Konig's Theorem*:

If G is bipartite, then the size of the max matching is equal to the size of the min cover

We also covered an algorithm for finding a max matching in a bipartite graph:

- Let M be any matching
- Construct X, Y for M
- If there exists an unsaturated vertex in Y, then there exists an augmenting path of M. Flip M along P to get a larger matching M'. Set $M = M'$ and go back to the previous step
- If not, then M is a max matching and $C = (A - X) \cup Y$ is a min cover

(This algorithm won't be on assignment 6, but it will be on later assignments and the exam)

20.1 Applications of Konig's Theorem

Question 1

If G is a bipartite graph, then when does G have a perfect matching?

The answer is: as long as one bipartition A is completely saturated and the other bipartition is the same size as A, then we have a perfect matching.
A more general question is:

If $G = (A, B)$ is a bipartite graph, when does G have a matching saturating all vertices of A?

To answer this second question, we need to outline an obvious necessary condition:

$$\forall S \subseteq A, |S| \leq |N(S)|$$

In other words, the number of vertices in every subset of A must be less than or equal to the number of neighbours in B connected to all the vertices in S.
This condition is in fact all that is required for A to have a perfect matching. This is known as **Hall's Theorem:**

20.1.1 Hall's Theorem

Let $G = (A, B)$ be bipartite. G has a matching saturating all vertices of A $\iff$ $\forall S \subseteq A, |S| \leq |N(S)|$

Proof of Hall's Theorem

- ($\Longrightarrow$): if there exists such a matching and $S \subseteq A$, then the neighbours of S in M have size S. Hence $|S| \leq |N(S)|$.

- ($\Longleftarrow$): recall from Konig's theorem that |max matching| = |min cover|. Now it suffices to show that G has a matching of size A (i.e., |max matching| $\geq |A|$). Thus, it suffices to show that |min cover| $\geq |A|$ (i.e., for every cover C, $|C| \geq |A|$).
  Claim: if C is a cover, then $|C| \geq |A|$.
  Proof: since C is a cover, there is no edge from $A - C$ to $B - C$. That means that every neighbour of a vertex in $A - C$ is in $B \cap C$. Let $S = A - C$. Thus, $N(S) \subseteq B \cap C$. Yet, by assumption, $|S| \leq |N(S)|$, $\forall S \subseteq A$. Thus

$$
\begin{aligned}
|S| &\leq |N(S)| \\
|A - C| &\leq |B \cap C| && \text{(Equal sets)}
\end{aligned}
$$

  Well, then

$$|C| = |A \cap C| + |B \cap C| \geq |A \cap C| + |A - C| = |A|$$

20.1.2 Corollary 1

Let $G = (A, B)$ is bipartite. G has a perfect matching $\iff$ $\forall S \subseteq A, |S| \leq |N(S)|$ and $|A| = |B|$

Proof of Corollary 1

Obviously the two conditions are necessary. By Hall's theorem, there exists a matching saturating A if the first condition is satisfied ($|S| \leq |N(S)|$), but then it saturates B if the second condition holds ($|A| = |B|$).

20.1.3 Corollary 2

Let $G = (A, B)$ be bipartite. If $\forall v \in A$, deg$(v) \geq k$, and $\forall v \in B$, deg$(v) \leq k$, then G has a matching saturating all of A

Proof of Corollary 2

By Hall's theorem, it suffices to check that $\forall S \subseteq A, |S| \leq |N(S)|$. So let $S \subseteq A$. Consider $E(S, N(S))$ that is the set of edges with one end in S and the other in $N(S)$. Since $\forall v \in A$, $\deg(v) \geq k$,

$$|E(S, N(S))| \geq k|S|$$

Yet since $\forall v \in B$, $\deg(v) \leq k$, we have

$$|E(S, N(S))| \leq k|N(S)|$$

Together, $k|S| \leq |E(S, N(S))| \leq k|N(S)|$ So, $|S| \leq |N(S)|$, as desired.

20.1.4 Corollary 3

(Prof. Postle calls this a pretty theorem)

If G is a k-regular ($k \geq 1$) bipartite graph, then G has a perfect matching

Proof

$\forall v \in A$, $\deg(v) = k \geq k$. And similarly, $\forall v \in B$, $\deg(v) = k \leq k$. Thus, by the previous corollary, there exists a matching saturating all of A. But then it suffices to show $|A| = |B|$. This follows form the handshaking theorem for bipartite graphs:

$$k|A| = \sum_{v \in A} \deg(v) = |E(A, B)| = \sum_{v \in B} \deg(v) = k|B|$$

— And this concludes the topic of graph theory in this course. After reading week we will begin combinatorics —