MATH 239 — Lecture 2

Bartosz Antczak *Instructor: Luke Postle* *January 6, 2017*

Review of last lecture

A **graph** is a set of of elements called **vertices** with a set of distinct pairs of vertices called **edges**. Some examples of graphs we've covered include:

- Complete (K_n)
- Cycle (C_n)
- Path (P_n)
- Complete bipartite graph ($K_{m,n}$, also called *cliques*)

2.1 Notation and terminology

$V(G)$ denotes the set of vertices on a graph G, and $E(G)$ denotes the set of edges. If $\{uv\} \in E(G)$ (NOTE: we can also denote $\{uv\}$ without the brackets, simply as uv), then we say u is adjacent to v, and we also say that v is a neighbour of u.

If $v \in V(G)$, we let $N(v)$ denote the neighbourhood of v, which is the set of neighbours of v. The number of neighbours of v, which is $|N(v)|$, is called the degree of v.

If $e = uv$ is an edge of G, then we say that u and v are the ends of e. Furthermore, we say e is incident with u or v and similarly u or v is incident with e. We also say two edges e_1 and e_2 are incident if they have a common end.

2.1.1 Definition: subgraph

A **subgraph** H of G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, or equivalently, H can be obtained from G by deleting some vertices (and all incident edges) and some additional edges.

Example 2.1.1. *$\forall\, m \le n$, K_m is a subgraph of K_n. Actually, every graph on at most n vertices is a subgraph of K_n*

Also, the only subgraph of C_n is C_n itself! What about P_m? P_m is a subgraph of N_n, $\forall\, m \le n$.

Deleting edges and vertices

If $v \in V(G)$, we let $G - v$ denote the graph obtained from G by deleting v and all incident edges.

If $e \in E(G)$, we let $G - e$ denote the graph obtained from G by deleting e.

2.2 Graph properties

We define graphs based on their properties, some examples include:

- Having a certain number of vertices (or edges)
- Having a vertex of degree 1
- Containing a triangle as a subgraph
- Having no cycle as a subgraph

One property we'll focus on is whether a graph is *bipartite*.

2.2.1 Definition: bipartite

A graph G is **bipartite** if there exists a partition of $V(G)$ into two disjoints A and B such that every edge of G has exactly one end in A and the other in B. Some examples include:

- $K_{m,n}$, $\forall\, m, m$
- P_z, $\forall\, z$
- K_1 and K_2; however, K_3 isn't bipartite! Let's prove it:

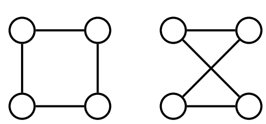
Proof

Proof by contradiction.
Suppose that K_3 is bipartite. This means that there exists a partition A, B of $V(G)$ with all edges with one in A and other in B. Note that at least one of A or B has size at least 2, because of the *pigeonhole principle*[1].
Now suppose without loss of generality (abbreviated to "wlog"), that $|A| = 2$. Let $u, v \in A$. Since K_3 is complete, $uw \in E(G)$ with both ends in A — a contradiction.

So what other graphs are not bipartite? Well, C_n where n is odd is not bipartite. Actually, any graph that contains a triangle is not bipartite. Why? Well it's because of a proposition: if H is a subgraph of G and G is bipartite, then so is H.

Question to finish this lecture

How do we know that two graphs are the same (are equal)? For example, $C_4 = K_{2,2}$:



Notice that these graphs are visually different, but the definition of graphs doesn't concern how they look like. Why are they equal? Because there exists a *bijection* between the vertex sets. More on this in the next lecture!

[1] m pigeons into n holes. If $m > n$, then there exists a hole with at least 2 pigeons.