MATH 239 — Lecture 19

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19.1 More on Matchings

We asked a question last lecture: when is the size of a max matching equal to the size of a min cover? Today we will analyse that:

- For K_n, the max matching is $\lfloor \frac{n}{2} \rfloor$ and the min cover is $n-1$
- For C_{2k+1} (odd cycle), the max matching is k, and the min cover is $k+1$. If there isn't an odd cycle, we are bipartite and 2-colourable.
- For k triangles (disconnected triangles), the max matching is k, and the min cover is $2k$

Since the two values are not equal in an odd cycle, what if we considered graphs without one? Specifically a bipartite graph. Could they be equal for a bipartite graph? From this stems **Konig's Theorem**.

19.1.1 Konig's Theorem

If G is bipartite, then the size of the max matching of G is equal to the size of the min cover

Proof of Konig's Theorem

(We will construct a matching and cover of equal size)
Let A and B be the two bipartitions of G. Let M be a maximum matching. Also, let X_0 be the unsaturated vertices of A (X_0 is non-empty as otherwise M has size $|A|$ and yet A is a cover of size $|A|$, as desired). Let Z be the set of all vertices that are the end of an alternating path whose other end is in X_0.
Let $X = A \cap Z$, and $Y = B \cap Z$. Let $C = (A - X) \cup Y$. I claim that C is the cover of the same size as M, which I will prove.

- **Claim:** there is no edge with one end in X and the other end in $B - Y$.

  Proof of Claim

  Let $uv, u \in X$, and $v \in B - Y$, be such an edge. Since $u \in X$, there exists an alternating path P from a vertex of X_0 to u. Either $u \in X_0$ and P is one vertex or the last edge is in M. If $u \in X_0$, then uv is an alternating path from u to v and so $v \in Y$, a contradiction.
  If $u \notin X_0$, then either v is not matched to u in M and then $P' = P + uv$ is an alternating path from a vertex of X_0 to v, and so $v \in Y$, a contradiction. Otherwise, v is matched in M to u, but then v is on P already and so $P' = P - uv$ is an alternating path and $v \in Y$, a contradiction.
  It follows from the claim that every edge has an end in at least one of $A - X$ or Y. Thus, C is a cover!

- **Claim 2:** every vertex of $A - X$ is saturated.

Proof of Claim 2

If there exists an unsaturated vertex, then it would be in X_0 and so in X.

- **Claim 3:** every vertex of Y is saturated.

  Proof of Claim 3

  If there existed a vertex which was unsaturated, then there would exist an alternating path with one end in X_0 and othe other end would be unsaturated in Y. But then P is an augmenting path and so M is not maximum, a contradiction.

From these claims, we see that every vertex in C is saturated.

- **Claim 4:** there exists an edge $e = uv \in M$ such that $u \in Y$, and $v \in A - X$.

  Proof of Claim 4

  Since $u \in Y$, there exists an alternating path P from a vertex of X_0 to u. The last edge of P mus not be in M, by parity. But then $P' = P + uv$ is an alternating path from that vertex of X_0 to v. So $v \in X$, a contradiction.

Finally, we claim that $|C| = |E(M)|$, and so C is a min cover and M is a max matching, by our earlier lemma.
To see this, let M_1 be the set of edges of M with an end in Y and let M_2 be the set of edges of M with an end in $A - X$. By our last claim, these are disjoint sets. Since C is a cover, every edge of M is in either M_1 or M_2. Since every vertex of Y is saturated, $|Y| = |E(M_1)|$, and similarly, since every vertex of $A - X$ is saturated, $|A - X| = |E(M_2)|$. So,

$$|E(M)| = |E(M_2)| + |E(M_2)| = |Y| + |A - X| = |C|$$

As required (phew!).

19.1.2 Algorithm for Finding a Max Matching in a Bipartition

1. Start with a matching M.
2. Construct X, Y, C from X_0.
3. If there exists an unsaturated vertex in Y, then there exists an augmenting path P of M. Flip P on M and return to step 2.
4. Otherwise, M is a max and C is a min cover.

19.1.3 Algorithmic Questions

For a fixed k,

1. Does G have a matching of size $\geq k$?

- NP? Yes for all graphs.
- co$-NP$? Yes for bipartite graphs. Simply show a cover of size $k-1$.

2. Let M be a matching of G. Is M maximum?

   - NP? Yes for bipartite. Show a cover of the same size.
   - co$-NP$? Yes for all graphs. Simply show a larger matching or an augmenting path.