MATH 239 — Lecture 18

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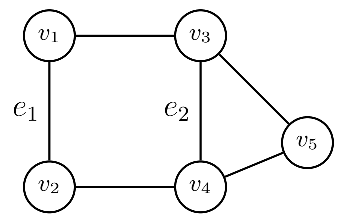
18.1 Matchings

The final topic for graph theory.
A **matching** in a graph is a set of edges such that no two edges in the matching are incident with a common vertex.

Example 18.1.1. *A graph and its labelled edges e_1 and e_2 that are in a matching*



Matchings can model trade deals, students and residencies, students and "slots" in classes, etc. Now we'll look at a couple of definitions.

Definition 1 — Maximum

A matching M or G is **maximum** if there is no larger matching of G.

Definition 2 — Saturated

A vertex v is **saturated** by a matching M if $\exists\, e \in M$ incident with v (**unsaturated** is the opposite). In example 18.1.1, v_1, v_2, v_3, v_4 are saturated; v_5 is unsaturated.

Definition 3 — Alternating Path

An **alternating path** P of M is a path in which every other edge is in M. In example, 18.1.1, v_1v_2, v_2v_4, v_4v_3 is an alternating path.

Definition 4 — Augmenting Path

An **augmenting path** P of M is an alternating path whose ends are unsaturated. This means that the first and last edge of an augmenting path are not in M. This implies that the number of edges in an augmenting path are *odd*.

18.1.1 Proposition 1

$$\textit{If } P \textit{ is augmenting, then } |E(P) \cap E(M)| < |E(P) - E(M)|$$

18.1.2 Proposition 2

If P is an augmenting path of a matching M, and M' is obtained from M by switching the edges of P in M for the edges of P not in M, then M' is a matching

Proof of Proposition 2

Every vertex of P is in at most one edge of M', since the ends are unsaturated in M.

18.1.3 Lemma 1

If M is a matching and P is an augmenting path of M, then M is not a maximum matching

Proof of Lemma 1

Let M' be obtained from M by switching on P. By proposition 2, M' is matching, yet $|E(P) \cap E(M)| < |E(P) - E(M)|$. Which leads to

$$|E(M)| = |E(M) \cap E(P)| + |E(M) - E(P)| < |E(M) - E(P)| + |E(P) - E(M)| = |E(M'|$$

So M' is larger than M and so M is not maximum.
Actually, the converse of lemma 1 is also true (proof omitted), so the statement is actually an if-and-only-if:

18.1.4 Lemma 1 (complete)

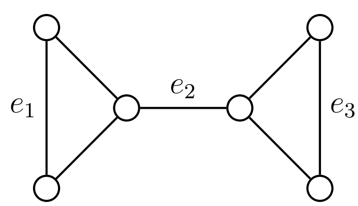
If M is a matching and P is an augmenting path of M if and only if M is not a maximum matching

This implies that deciding if a matching is max is in co$-NP$. So we have a tool to show a matching is not maximum, but do we have a tool to show that is it?

Definition 5 — Perfect Matching

A **perfect matching** is a matching saturating every vertex. **Note:** not every graph has a perfect matching.

Example 18.1.2. *A graph with a perfect matching (edges in the matching are labelled e_i)*

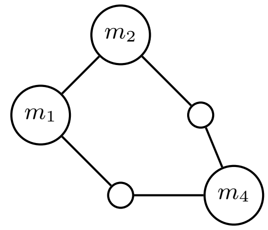


Our final tool for the day:

Definition 6 — Cover

A **cover** is a set of vertices such that every edge has at least one end in the cover (it's called a *cover* because it represents a set of vertices that <u>cover</u> all of the edges).

Example 18.1.3. *A graph with a cover (the vertices in the cover are labelled m_i)*



18.1.5 Lemma 2

If M is a matching and C is a cover, then $|E(M)| \leq |V(C)|$

Proof of Lemma 2

The cover C has to cover all the edges of M (i.e., $\forall\, e \in M$, at least one end must be in C). So C has at least $|E(M)|$ vertices since no two edges of M share a vertex, as desired.

18.1.6 Corollary 1

The size of the max matching is less than the size of a min cover

Proof of Corollary 1

If M is a matching and C is a cover, by lemma 2, $|M| \leq |C|$.

Corollary 2

If M is a matching and C is a cover and $|M| = |C|$, then M is a max matching and C is a min cover

Proof of Corollary 2

There can't be a larger matching M' because then $|M'| > |M| = |C|$, a contradiction. Similarly, there can't be a smaller cover C' because then $|C'| < |C| = |M|$, a contradiction.

A question we'll look at next lecture is: when is the max matching equal to the min cover