MATH 239 — Lecture 17

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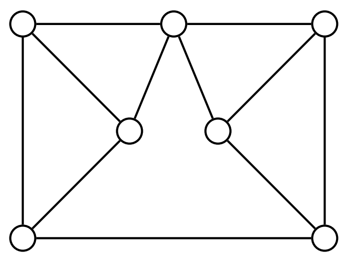
17.1 More on Colouring

Definition — D-degenerate

A graph G is **d-degenerate** is every subgraph H of G has a vertex of degree at most d in H.

Example 17.1.1. *Consider the following graph. The max degree found in every subgraph H of G is at most 3 (observe that there is a vertex of degree 4, but this vertex isn't found in every subgraph).*



An equivalent formulation is:

G is **d-degenerate** if there exists an ordering $v_1, v_2, \cdots, v_n \in V(G)$ such that for all i, the number of neighbours of v_i with index $< i$ is at most d.

These definitions are equivalent.

17.1.1 Lemma 1

(Will be useful for the homework)

If G is $d-$degenerate, then G is $(d+1)-$colourable

Proof of Lemma 1

Since G is $d-$degenerate, there exists an ordering $v_1, \cdots, v_n$ as outlined in the definition previously mentioned. Now colour the vertices of G in that order, giving v_i the lowest colour in $\{1, \cdots, d+1\}$ not used by its earlier neighbours. Thus, we have $d+1$ colours.

17.1.2 Question 1

How many colours (in the worst case) do you need to colour a planar graph? More formally, which is the maximum value of $\chi(G)$ for all planar G?

17.1.3 Theorem 1 (6 colour theorem)

If G is a planar graph, then G is 6-colourable

This follows from our next lemma, combined with Lemma 1.

17.1.4 Lemma 2

If G is planar, then G is 5-degenerate

Proof of Lemma 2

Let H be a subgraph of G. Since G is planar, then H is planar. By our corollary of Euler's formula,

$$|E(H)| \leq 3|V(H)| - 6$$

By the Handshaking lemma

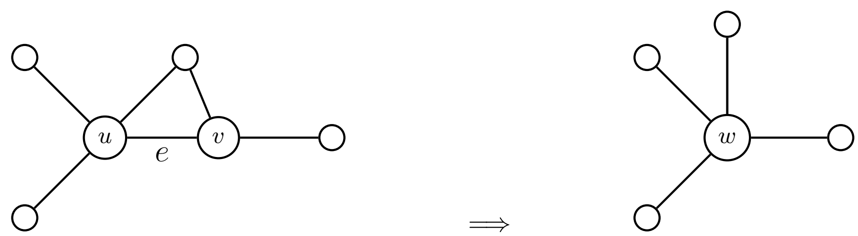
$$\sum_{v \in V(H)} \deg(v) = 2|E(H)| \leq 6|V(H)| - 12$$

Thus, there exists a vertex of H of degree at most 5, otherwise, there exists a degree of at least 6, which means that $\sum_{v \in V(H)} \deg(v) \geq 6|V(H)|$, which is a contradiction.

Definition — Contraction

If $e = uv$ is an edge of G, then the **contraction** of e, denoted G/e is the graph where we delete u and v and add a new vertex w adjacent to all of $N(u) \cup N(v)$ (N denotes the neighbours of the current vertex).

Example 17.1.2. *A contracted graph*



17.1.5 Proposition 1

If G is a planar graph and $e \in E(G)$, then G/e is planar

Proof is left as an exercise to the reader.

17.1.6 5-colour Theorem

If G is planar, then G is 5-colourable

Proof of 5-colour Theorem

We proceed by induction on $|V(G)|$. By our lemma, G is 5-degenerate and so has a vertex v of at most 5. Consider two cases:

- $\deg(v) \leq 4$: then colour $G - v$ with a 5-colouring by induction. Then extend the colouring to v by giving v a colour different from all of its neighbour's colours. This is possible because v has at most 4 neighbours and we have at least 5 colours.

- $\deg(v) = 5$: let x, y be non-adjacent neighbours of v. Note, such x, y exist as otherwise, $N(v) = K_5$, contradicting that G is planar. Now let $G' = G/e_{\{vx,vy\}}$ (i.e., contract two edges). Let z be the new vertex in place of x, v, y whose neighbourhood in G' is $N(x) \cup N(y) \cup N(v) - \{v, x, y\}$.
  By our proposition, G' is planar. Yet, $|V(G')| \leq |V(G)|$, thus my induction, G' has a 5-colouring. Now we extend this colouring to G by:
  - colouring both x and y with the colour of z, then
  - colour v with a colour not used by its neighbours. This works because we have at most 5 neighbours but two (x and y) have the same colour.

In fact, we can go even deeper. A planar graph can be in fact **4-colourable**.