MATH 239 — Lecture 16

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Review of Last Lecture

When we **subdivide** an edge $e = uv$, we delete e and add a new vertex w adjacent to only u and v.
An $H-$subdivision is a graph obtained from H by subdividing edges.
We also covered an if-and-only-if theorem about planar graphs:

$$G \textit{ is planar } \iff G \textit{ has no } K_5\textit{−substitution and } K_{3,3}\textit{−substitution}$$

To decide if G is planar, either give an embedding or give a K_5-subdivision or $K_{3,3}-$subdivision. So deciding planarity is in NP and co$-NP$, and actually it's in P. There is a linear-time algorithm (but it's complicated so we won't cover it).

16.1 Colouring

Prof. Postle wrote it as "coloring" because he's American, and he refuses to include a 'u' because "screw you".
Let's begin with a question:

Is there a natural generalization of a bipartite graph? (e.g., a tripartite graph?)

Recall that if G is bipartite if there exists a bipartition of $V(G)$ into two parts A and B such that no edge has both ends in the same partition.

Definition — Independent Set

An **independent set** I of G is a set of vertices with no edge with both ends in I. From this, we can construct another (but equivalent) definition of a bipartition:

G is bipartite if there exists a partition of G into two parts, each of which is an independent set.

From this, we can define a *tripartite* using the previous definition, except we swap the word "two" with "three". However, instead of this definition, we'll use different terminology:

Definition — K-colouring

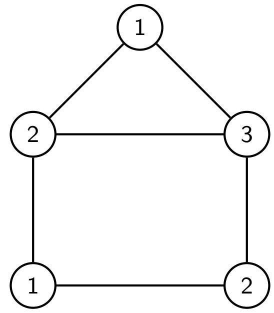
A **k-colouring** of a graph G is a partition of $V(G)$ into k independent sets.
Why do we use this terminology? We can attribute each independent set with a unique colour. This results in no edge having two ends of the same colour.

Definition — K-colourable

A graph G is $k-$colourable if there is a $k-$colouring of G.
From this, we create another theorem:

$$G \textit{ is bipartite } \iff G \textit{ is 2-colourable}$$

Example 16.1.1. *A 3-colourable graph, with partitions 1, 2, and 3*



Remark

If G is $k-$colourable, then G is $m-$colourable for all $m \geq k$ (because the independent sets may be empty).

Definition — Chromatic number

The **chromatic number** of a graph G denoted, $\chi(G)$, is the minimum k such that G is $k-$colourable.

16.1.1 Proposition 1

If H is a subgraph of G, then $\chi(H) \leq \chi(G)$

Proof of Proposition 1

Let $k = \chi(G)$ and consider a $k-$colouring of G, then that induces a $k-$colouring of H, so $\chi(H) \leq k$.

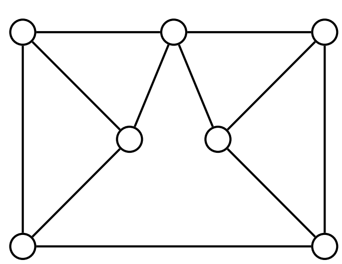
16.1.2 Proposition 2

If G contains k_n as a subgraph, then $\chi(G) \geq n$

Definition — Clique Number

Denoted $w(G)$, is the size of a largest complete subgraph of G. $\chi(G) \geq w(G)$. This is the *lower bound* on determining the chromatic number of a graph G (recall that this is a greater-than-or-equal-to sign, so the clique number isn't always the chromatic number of a graph G).

Example 16.1.2. *For this graph, $\chi(G) = 4$. Can you match each vertex into their respective partition?*



Is deciding if G is $k-$colourable in NP? P? co$-NP$? It's actually in all three!

Definition — Maximum Degree

The **maximum degree** of a graph G, denoted $\Delta(G)$, equals $\max_{v \in V(G)} \deg(v)$

16.1.3 Theorem 1

$$\chi(G) \leq \Delta(G) + 1 \textit{ for all graphs } G$$

Proof of Theorem 1

Let $v_1, v_2, \cdots, v_n$ be an arbitrary ordering of $V(G)$. Now colour the vertices in order, where we colour v_i with the lowest colour in $\{1, \cdots, \Delta(G) + 1\}$ that's not used by any neighbours. Such a colour exists because $\deg(G) \leq \Delta(G)$ but there are $\Delta(G) + 1$ colours.