MATH 239 — Lecture 15

Bartosz Antczak *Instructor: Luke Postle* *February 6, 2017*

Review of Last Lecture

A **platonic solid** is a plane graph where all vertices have the same degree d and all faces have the same degree d^*. A **theorem** we discussed was

There are exactly five platonic solids: the tetrahedron, cube, octahedron, icosahedron, dodecahedron

(A visual of these platonic solids was shown last lecture). Additionally, we covered a **lemma**:

If G is a platonic solid, then $(d, d^*) \in \{(3,3),(3,4),(4,3),(3,5),(5,3)\}$

Now recall:

$$2E = d^*F = dV \qquad \text{(By handshaking + handshaking for faces)}$$

By Euler's formula:

$$\begin{aligned} &\implies V - E + F = 2 \\ &\implies \frac{2E}{d} - E + \frac{2E}{d^*} = 2 \\ &\implies E = \frac{2dd^*}{2(d+d^*) - dd^*} \qquad \text{(Solving for } E\text{)} \end{aligned}$$

15.1 Proof of there being only 5 Platonic Solids

To prove that there are only five platonic solids, we must show that the only solid with a particular (d, d^*) pairing, then it's the *only* one in our list. We consider 5 cases (we'll prove only the first two):

15.1.1 Lemma 1

If G is a platonic solid with $d = d^* = 3$*, then G is the tetrahedron*

Proof of Lemma 1

By calculation, $V = 4$, and $E = 6$. But then G is K_4, which only has one planar embedding (and $F = 4$) (i.e., G is the tetrahedron).

15.1.2 Lemma 2

If G is a platonic solid with $d = 3, d^* = 4$*, then G is the cube*

Proof of Lemma 2

By calculation, given our (d, d^*), $V = 8$, $E = 12$, and $F = 6$. Let G be such a solid. Let f be a face of G. Note that f has degree 4 and so is a 4-cycle $C = v_1v_2v_3v_4$. We claim that v_1 is not adjacent to v_3. Suppose not, then the other face f' containing v_1v_4 must also contain v_1v_3, v_4v_3, and v_4w (twice) where w is the

other neighbour of v_4. But then $\deg(f') = 5$, a contradiction. This proves the claim that v_1 is not adjacent to v_3. By symmetry, v_2 is not adjacent to v_4.
Let u_i be the other neighbour of v_i. Now I claim that all u_i are distinct. To prove this, suppose not. There are 2 cases up to symmetry:

- **Case 1:** $u_1 = u_2$. But then the other face containing v_1v_2 is either a triangle or has degree greater than or equal to 5. a contradiction.
- **Case 2:** $u_1 = u_3$. But then, u_2 must be adjacent to $u_1 = u_3$ for the face containing v_1v_2 to be a 4-cycle. But then u_2 has degree 2, a contradiction.

But now $u_iu_{i+1(\text{mod}4)}$ is an edge since the face containing $v_iv_{i+1(\text{mod}4)}$ is a 4-cycle. But then G is the cube, as desired.

15.2 Let's Return to the Big Question

When is a graph G planar?
We begin by analysing if this decision problem is in either P, NP, or co$-NP$. Observe that we can just check an embedding in polynomial time, which means that it's true for NP.
But what about co$-NP$? It's not obvious since there are exponentially many embeddings. It turns out that the answer is in fact *yes*, which will be proven by corollary 2 at the end of this lecture.
Recall:

If G is planar, then G does not have a non-planar subgraph (e.g., K_5 or $K_{3,3}$)

We'll begin with a definition:

Definition — Subdivide

If G is a graph and $e = uv \in E(G)$, then to **subdivide** e is to delete it and add a new vertex w adjacent to both u and v:

Definition — Subdivisions (Not the Rush song)

A **subdivision** of a graph H (also called an $H-$subdivision) is a graph obtained from H by subdividing edges (those edges had to be cool or be cast out).
We will use this definition determine when a graph is planar using the following lemma:

15.2.1 Lemma 3

Let H be a subdivision of G. Then,

$$H \text{ is planar} \iff G \text{ is planar}$$

Proof of Lemma 3

- **($\Longleftarrow$):** if G is planar, then just subdivide its planar embedding to get a planar embedding of H.
- **($\Longrightarrow$):** if H is planar, then in a planar embedding, replace the paths of degree 2 with the original edges of G to get a planar embedding of G (i.e., 'unsubdivide').

15.2.2 Corollary 1

If G is planar, then G does not contain (as a subgraph) a K_5-subdivision or $K_{3,3}-$subdivision

So not having a K_5-subdivision or a $K_{3,3}-$subdivision is a necessary condition to be planar. Surprisingly, it's also a sufficient condition. This leads to Kuratowski's theorem.

15.2.3 Kuratowski's Theorem

$$G \text{ is planar} \iff G \text{ contains no } K_5-\text{subdivision or } K_{3,3}-\text{subdivision}$$

15.2.4 Corollary 2

Deciding planarity is in co$-NP$

Proof of Corollary 2

Simply show the K_5 or $K_{3,3}-$subdivision.