MATH 239 — Lecture 12

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Review of Last Lecture

We defined a **weighted graph**, which is a graph with a weight function on its edges. From this, we discussed an **MST** (Minimum Spanning Tree), which is a spanning tree of a weighted graph which has minimum weight over all spanning trees (where weight of $T = \sum_{e \in E(T)} w(e)$. Finally, we mentioned **Prim's Algorithm** and proved the theorem that:

The output of Prim's algorithm in an MST

Prof. Postle outlined two parts in the proof that may have been confusing and proved them as lemmas:

Lemma 1

If T is a spanning tree and $e \notin E(T)$, then $\forall f$ in the unique cycle of $T+e$, $T' = T+e-f$ is a spanning tree

Proof: $T+e$ is connected because T is. Now, $T' = T+e-f$ is connected because f is in a cycle of $T+e$ and so it's not a bridge. So T' has the same number of components as $T+e$, that is, one.
So T' is connected, and T' has no cycles since we deleted the edge f, which was in the only cycle. So T' is a tree and is clearly spanning.

Lemma 2

Let C be a cycle of G and X is a proper, non-empty subset of $V(G)$. Then $|E(C) \cap \delta(X)|$ is even

Proof (sketch): Direct C and pair the edges as they go in/out of X.

12.1 Planar Graphs

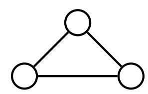
The prettiest topic of Graph Theory, according to Prof. Postle.
A graph is **planar** if it can be drawn in the plane $\mathbb{R}^2$ such that edges go between their ends and edgs do not otherwise intersect (i.e., do not "cross"). We call such a drawing a planar embedding of G:

Example 12.1.1. *A non-planar embedding of a graph (left) and its planar embedding (right)*

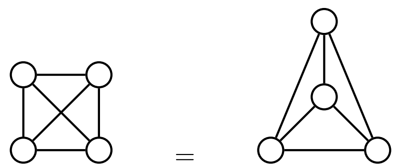
12.1.1 The Big Question with Planar Graphs

Which graphs have a planar embedding?

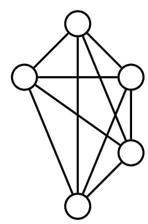
Example 12.1.2. *K_3 is planar*



Example 12.1.3. *K_4 is also planar (we can move edges and bend them)*



All cycles, paths, and trees are planar. But what about this graph:



This graph, K_5, is *not* planar. No matter how you draw it, there will always be a crossing. And actually, once you know K_5 is not planar, then K_n is not planar for all $n \geq 5$. But why?

Proposition 1

If G is planar and H is a subgraph of G, then H is planar

Proof: finding any subgraph of G will involve deleting edges and vertices from the planar embedding of G. Since no edges were crossed during this process, every subgraph must also be planar.

Corollary 1

If G contains a non-planar subgraph H, then G is also non-planar

Proof: this is the contrapositive of proposition 1.

12.1.2 Finishing off the Lecture

Corollary 1 proves the previous statement where K_n is non-planar for all $n \geq 5$.
What about complete bipartite graphs ($K_{m,n}(m \geq n)$)? If m or n are less than 2, then it's planar.
What about $K_{3,3}$? It's *not* planar.

$K_{3,3}$ and K_5 are the only "reasons" a graph is not planar. In other words, if you can find $K_{3,3}$ or K_5 as a subgraph in a graph G, then G is non-planar.