MATH 239 — Lecture 10

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Review of Last Lecture

An **Eulerin circuit** is a closed walk using every edge exactly once. A theorem on this is:

A graph G has an Eulerian circuit if and only if every vertex of G has even degree and all edges lie in the same component of G

Hence, it is easy to decide if G has an Eulerian circuit. If G has one, use the following algorithm to find it:

- Let $v \in V(G)$ with degree greater than or equal to 2
- Find a closed walk W containing v using no edge more than once
- While $E(W) \neq E(G)$:
  - Find a vertex $u \in V(W)$ incident to an edge not in W
  - Find a closed walk W' in $G' = G - E(W)$
  - Insert W' into W at vertex u

Definition - Eulerian Path

An **Eulerian path** is a *walk* (pretty counter-intuitive) that uses every edge exactly once.
Question 1: *When does G have an Eulerian path?*
Remark: *If you have an Eulerian circuit, then you have an Eulerian path*

Question 2: *Are there graphs that don't have an Eulerian circuit but have an Eluerian path?*
Answer: *Yes, all paths P_k!*
Some necessary conditions for an Eulerian path include:

- At most 2 odd degree vertices (either 0 or 2), because only the end of the walk can be odd
- All edges lie in the same component

Theorem 9.0.1

G has an Eulerian path if and only if G has at most 2 odd degree vertices and all edges lie in the same component

Proof of Theorem 9.0.1:

- **Proof of ($\Longrightarrow$)**: by observation, this is true.
- **Proof of ($\Longleftarrow$)**: if G has 0 odd degree vertices, then G only has even-degreed vertices and by Euler's Theorem, G has an Eulerian circuit, and thus an Eulerian path.
  If G has exactly two odd degree vertices, call them u and v. Let G' be obtained from G by adding a new vertex w adjacent to u and v. Now in G', all edges lie in the same component and all vertices in G' have even degree. So by Euler's Theorem, there exists an Eulerian circuit W, but $W' = W - \{uw, wv\}$ is an Eulerian path in G.

10.1 Characterizing Bipartite Graphs

Recall that a graph G is *bipartite* if there exists a partition A and B of $V(G)$ such that no edge has both ends in the same partition.

Proposition 1

If G has an odd cycle, then G is not bipartite

This statement holds because odd cycles are not bipartite and every subgraph of a bipartite graph is bipartite. Today, let's prove the converse:

If G has no odd cycles, then G is bipartite

Definition — Distance

The **distance** between two vertices u and v, denoted $d(u, v)$ is the length (in edges) of a shortest path between u and v. Some axioms of this definitions include:

- $d(u, u) = 0 \quad \forall u$
- $d(u, v) = 1 \iff uv \in E(G)$

Proposition 2

Every tree is bipartite

Proof: let T be a tree and $r \in V(T)$. Now let $A = \{u \in V(T) : d(r, u) \text{ is even}\}$ and $B = \{u \in V(T) : d(r, u) \text{ is odd}\}$. Now every edge has one end in A and the other in B. Thus, (A, B) is a bipartition.

Proposition 3

A graph is bipartite if and only if all of its components are bipartite

Proof of ($\implies$): if G is bipartite, then it has a bipartition (A, B) with every edge between A and B. Let H be a component of G, but then $(A \cap V(H), B \cap V(H))$ is a bipartition of H, as desired.
Proof of $\impliedby$: let $G_1, G_2, \cdots, G_k$ be components of G. Let (A_i, B_i) be a bipartition of G_i. Then $(\cup_i A_i, \cup_i B_i)$ is a bipartition of G, as desired.

Theorem 9.1.1

If G has no odd cycles, then G is bipartite

Proof: since G is connected, G has a spanning tree T. By proposition, T has a bipartition (A, B) such that all edges of T have one end in A and the other in B, as desired. I claim that (A, B) is a bipartition such that all edges of G have one end in A and one end in B.
I'll prove this claim by contradiction. Suppose there exists an edge $e = uv$ with both ends in the same partition. But then e is not in $E(T)$, since (A, B) is a bipartition of T. We may assume without loss of generality that $u, v \in A$. Now, let $P = x_1 x_2 \cdots x_k$ be a path in T where $x_1 = u$ and $x_k = v$.

We now claim that x_k is odd: note that $x_1 = u$ is in A, but then x_2 is in B because $x_1x_2 \in E(T)$. So inductively x_i is in A if i is odd and in B if i is even. But $x_k = v$ is in A, so k is odd.
Back to our main claim: let $C = P + e = x_1x_2 \cdots x_kx_1$ is an odd cycle in G, a contradiction. This means that (A, B) is a bipartition such that all edges of G have one end in A and the other in B, which proves this statement.